# Computer Science 331

## Merge Sort

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Lecture #25



## Outline







## Introduction

Merge Sort is is an asymptotically faster algorithm than the sorting algorithms we have seen so far.

- It can be used to sort an array of size $n$ using $\Theta(n \log_2 n)$ operations in the worst case.

Presented here: A version that takes an input array $A$ and produces another sorted array $B$ (containing the entries of $A$, rearranged)

A solution to the "Merging Problem" (presented next) is a subroutine that is used to do much of the work.





## The "Merging" Problem

*Calling Sequence:* `void merge(int [] `$A_1$`, int [] `$A_2$`, int [] `$B$`)`

*Precondition:*

- $A_1$ is a sorted array of length $n_1$ (positive integer) such that
$$A_1[h] \leq A_1[h+1] \qquad \text{for } 0 \leq h \leq n_1 - 2$$
- $A_2$ is a sorted array of length $n_2$ (positive integer) such that
$$A_2[h] \leq A_2[h+1] \qquad \text{for } 0 \leq h \leq n_2 - 2$$
- Entries of $A_1$ and $A_2$ are integers (more generally, objects from the same ordered class)

## The "Merging" Problem (cont.)

*Postcondition:*

- $B$ is a sorted array of length $n_1 + n_2$, so that

$$B[h] \leq B[h+1] \qquad \text{for } 0 \leq h \leq n_1 + n_2 - 2$$

- Entries of $B$ are the entries of $A_1$ together with the entries of $A_2$, reordered but otherwise unchanged
- $A_1$ and $A_2$ have not been modified





## Idea for an Algorithm

Maintain indices into each array (each initially pointing to the leftmost element)

**repeat**

- Compare the current elements of each array
- Append the smaller entry onto the "end" of $B$, advancing the index for the array from which this entry was taken

**until** one of the input arrays has been exhausted

Append the rest of the other input array onto the end of $B$





## Pseudocode

```
void merge(int [] A1, int [] A2, int [] B)
  n1 = length(A1); n2 = length(A2)
  Declare B to be an array of length n1 + n2
  i1 = 0; i2 = 0; j = 0
  while (i1 < n1) and (i2 < n2) do
    if A1[i1] ≤ A2[i2] then
      B[j] = A1[i1]; i1 = i1 + 1
    else
      B[j] = A2[i2]; i2 = i2 + 1
    end if
    j = j + 1
  end while
```



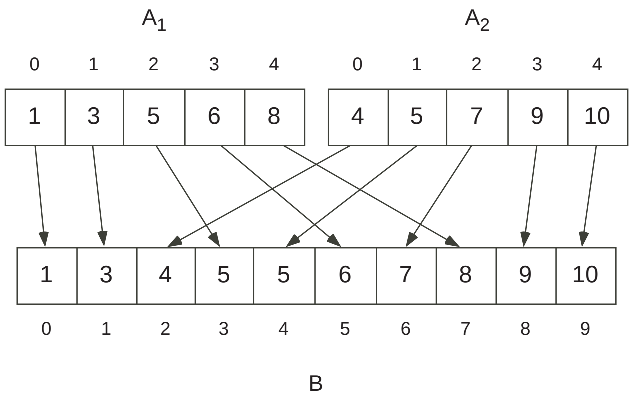


## Pseudocode, Continued

```
{Copy remainder of A1 (if any)}
while i1 < n1 do
  B[j] = A1[i1]; i1 = i1 + 1; j = j + 1
end while

{Otherwise copy remainder of A2}
while i2 < n2 do
  B[j] = A2[i2]; i2 = i2 + 1; j = j + 1
end while
```

# Example




**Note:** Running time is $\Theta(n_1 + n_2)$, where the input arrays have size $n_1$ and $n_2$





# Merge Sort: Idea for an Algorithm

Suppose we:

1. Split an input array into two roughly equally-sized pieces.
2. *Recursively* sort each piece.
3. Merge the two sorted pieces.

This sorts the originally given array.

Note: this algorithm design strategy is known as *divide-and-conquer*:

- divide the original problem (sorting an array) into smaller subproblems (sorting smaller arrays)
- solve the smaller subproblems *recursively*
- combine the solutions to the smaller subproblems (the sorted subarrays) to obtain a solution to the original problem (merging the sorted arrays)





# Pseudocode

`void` **mergeSort**(`int []` $A$, `int []` $B$)

  $n = A.length$
  **if** $n == 1$ **then**
    $B[0] = A[0]$
  **else**
    $n_1 = \lceil n/2 \rceil$
    $n_2 = n - n_1$ {so that $n_2 = \lfloor n/2 \rfloor$}
    Set $A_1$ to be $A[0], \ldots, A[n_1 - 1]$ {length $n_1$}
    Set $A_2$ to be $A[n_1], \ldots, A[n-1]$ {length $n_2$}
    **mergeSort**($A_1$, $B_1$)
    **mergeSort**($A_2$, $B_2$)
    **merge**($B_1$, $B_2$, $B$)
  **end if**





# Example

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $A$ : | 7 | 3 | 9 | 6 | 5 | 2 | 1 | 8 |

1. Sort $A[0, \ldots, 3] = [7, 3, 9, 6]$ recursively:
   - Sort $A[0, 1] = [7, 3]$ recursively
     - Sort $A[0] = [7]$ recursively — base case
     - Sort $A[1] = [3]$ recursively — base case
     - Merge: result is $[3, 7]$
   - Sort $A[2, 3] = [9, 6]$ recursively. Result is $[6, 9]$
   - Merge: result is $[3, 6, 7, 9]$
2. Sort $A[4, \ldots, 7] = [5, 2, 1, 8]$ recursively. Result is $[1, 2, 5, 8]$
3. Merge: result is $[1, 2, 3, 5, 6, 7, 8, 9]$

## Correctness of MergeSort

**Theorem 1**

*If* **mergeSort** *is run on an input array A of size $n \geq 1$, then the algorithm eventually halts, producing the desired sorted array as output.*

Prove by (strong) induction on $n$ (assuming that **merge** is correct!):

Base Case: $n = 1$

- if $n = 1$, array consists of one element (array is sorted trivially)
- algorithm returns $B$ containing a copy of the single element in the array (terminates with correct output)





## Correctness, continued

Inductive hypothesis:

- assume the algorithm is correct for input arrays of size $k < n$

Let $A$ be an array of length $n \geq 2$. Prove that $B$ is sorted copy of $A$.

- $A_1$ contains first $n_1$ elements of $A$
- $A_2$ contains remaining $n_2$ elements of $A$
- $n_1 = \lceil n/2 \rceil < n$ and $n_2 = \lfloor n/2 \rfloor < n$, so inductive hypothesis implies that $B_1$ is $A_1$ sorted and $B_2$ is $A_2$ sorted
- **merge** computes $B$ containing all elements of $A$ sorted (assuming that **merge** is correct)
- hence, algorithm is partially correct by induction. $\square$





## Termination and Efficiency

Let $T(n)$ be the number of steps used by this algorithm when given an input array of length $n$, in the worst case.

We can see the following by inspection of the code:

$$T(n) \leq \begin{cases} c_0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + c_1 n & \text{if } n \geq 2 \end{cases}$$

for some constants $c_0 > 0$ and $c_1 > 0$.

*Useful Relation:*

- For any integer $n$: $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.





## Termination and Efficiency, continued

*Recall That:*

- Operators $\lceil \rceil$ and $\lfloor \rfloor$ do not change the asymptotic behavior.

$T(n)$ can be rewritten as follows:

$$T(n) \leq \begin{cases} c_0 & \text{if } n = 1 \\ 2T(n/2) + c_1 n & \text{if } n \geq 2 \end{cases}$$

## Termination and Efficiency, continued

Recurrence Substitution for $n \geq 2$:

$$\begin{aligned} T(n) &\leq 2T(n/2) + c_1 n \\ &\leq 2\left(2T(n/2^2) + c_1 n/2\right) + c_1 n \\ &= 2^2 T(n/2^2) + 2c_1 n \\ &\leq \cdots \\ &\leq 2^k T(\frac{n}{2^k}) + kc_1 n \end{aligned}$$

Termination: $\frac{n}{2^k} = 1 \implies k = \log_2 n$





## Termination and Efficiency, concluded

**Theorem 2**

$$T(n) \leq nc_0 + (n \log_2 n)c_1.$$

Prove by induction on $n$

- Base case ($n = 1$)
- Inductive Step ($n \geq 2$)

**Conclusion:** Worst-case is in: $O(n \log_2 n)$





## Further Observations

It can be shown (by consideration of particular inputs) that the worst-case running time of this algorithm is also in $\Omega(n \log_2 n)$. It is therefore in $\Theta(n \log_2 n)$.

- This is preferable to the classical sorting algorithms, for sufficiently large inputs, if worst-case running time is critical.
- The classical algorithms are *faster* on sufficiently *small* inputs because they are simpler.

*Alternative Approach:* A "hybrid" algorithm:

- Use the recursive strategy given above when the input size is greater than or equal to some (carefully chosen) "threshold" value.
- Switch to a simpler, nonrecursive algorithm (that is faster on small inputs) as soon as the input size drops to below this "threshold" value.





## References

Further Reading:

**Introduction to Algorithms**, Chapter 2.3