Computer Science 331

Proofs of Tree Properties

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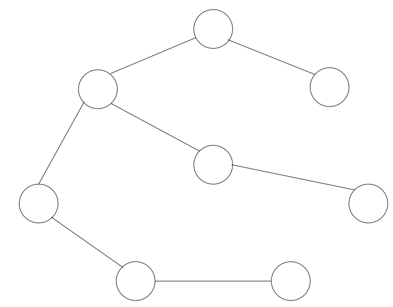
Lecture #31 Supplement

Trees

Definition: A *free tree* is a connected acyclic graph.



Frequently we just call a free tree a "tree."

- If we identify one vertex as the "root," then the result is the kind of "rooted tree" we have seen before.

Properties

We will present various properties and relations between $|V|$ and $|E|$ that characterize trees. Examples:

- If G is a tree then it has $|V|-1$ edges
- An acyclic graph with $|V|-1$ edges is a tree
- A connected graph with $|V|-1$ edges is a tree

Existence of Vertex With Degree At Most 1

Lemma 1

If $G=(V,E)$ is a graph such that $|V| \geq 2$ and $|E| < |V|$ then there exists a vertex $v \in V$ whose degree $d(v) \leq 1$.

Proof (by contradiction).

For any graph G, $\sum_{v \in V} d(v) = 2|E|$ (each edge counted twice)

If $d(v) \geq 2$ for every $v \in V$, then

$$2|E| = \sum_{v \in V} d(v) \geq \sum_{v \in V} 2 = 2|V|$$

so that $|E| \geq |V|$ — contradiction.

Thus, at least one vertex has degree at most one. □

Connected Graph has at Least $|V| - 1$ Edges

Lemma 2

If $G = (V, E)$ is connected then $|E| \geq |V| - 1$.

Proof (of contrapositive by induction on V).

Base case ($|V| = 0, 1$): G is connected, and $|E| = 0 \geq |V| - 1$

Contrapositive: If $|E| < |V| - 1$ then G is not connected

Suppose $|V| \geq 2$ and $|E| < |V| - 1$. By Lemma 1, $\exists v$ with $d(v) \leq 1$.

1. If $d(v) = 0$: G is not connected (v has no edges)
2. If $d(v) = 1$: let $G' = (V', E')$ be obtained by removing v and its one edge (so $|E'| = |E| - 1$ and $|V'| = |V| - 1$).
   - $|E'| < |V'| - 1$, and by the induction hypothesis G' is not connected.
   - G is also not connected (adding vertex and one incident edge). □

Property of Cyclic Graphs

Lemma 3

If $G = (V, E)$ and each vertex $v \in V$ has degree at least two then G includes a cycle.

Proof.

Pick $v_1 \in V$, follow edges in E to reach $v_1, v_2, \ldots$ until either

1. some vertex appears for the second time, or
2. all edges incident to the current vertex have been used

Notice that:

- one of these cases must arise (because $|V|$ and $|E|$ are finite)
- if every $v \in V$ has $d(v) \geq 2$, then Case 1 occurs before Case 2

Thus, G includes a cycle. □

Acyclic Graph has at Most $|V| - 1$ Edges

Lemma 4

If $G = (V, E)$ is acyclic then $|E| \leq |V| - 1$.

Proof (of contrapositive by induction on $|V|$).

Contrapositive: If $|E| > |V| - 1$, then G has a cycle

Base case ($|V| = 1$): if $|E| > |V| - 1 = 0$, then v has a loop (cycle)

Inductive step: Suppose that $|V| \geq 2$ and $|E| > |V| - 1$.

1. If $\exists v \in V$ with $d(v) < 2$: $G' = (V', E')$ obtained by removing v and its edge (if $d(v) = 1$) has $|E'| > |V'| - 1$ and has a cycle by induction hypothesis (thus, so does G)
2. Otherwise ($d(v) \geq 2$ for all $v \in V$): result follows by Lemma 3. □

A Tree has $|V| - 1$ Edges

Corollary 5

If $G = (V, E)$ is a tree then $|E| = |V| - 1$.

Proof.

If G is a tree, then:

- G is connected, so $|E| \geq |V| - 1$ by Lemma 2
- G is acyclic, so $|E| \leq |V| - 1$ by Lemma 4

Therefore, $|E| = |V| - 1$ □

Acyclic Graph with $|V| - 1$ Edges is a Tree

Lemma 6

If $G = (V, E)$ is acyclic and $|E| = |V| - 1$ then G is a tree.

Proof (induction on $|V|$).

Base case ($|V| = 1$): if $|E| = |V| - 1 = 0$, then G is a tree (single vertex, no edges)

Suppose $|V| \geq 2$. By Lemma 1, $\exists v \in V$ with $d(v) \leq 1$. Let $G' = (V', E')$ be obtained by removing v and its edge.

- $d(v) \neq 0$: $|E'| = |E|$ and $|V'| = |V| - 1$, so $|E'| = |V'|$, and by Lemma 4, G' (and thus G) has a cycle. Contradiction!
- Thus, $d(v) = 1$: G' is acyclic and has $|E'| = |V'| - 1$. By the induction hypothesis G' is a tree, and thus so is G. □

Connected Graph with $|V| - 1$ Edges is a Tree

Lemma 7

If $G = (V, E)$ is connected and $|E| = |V| - 1$ then G is a tree.

Proof (induction on $|V|$).

Base case ($|V| = 1$): same as Lemma 6

Suppose $|V| \geq 2$. By Lemma 1, $\exists v \in V$ with $d(v) \leq 1$.

1. If $d(v) = 0$: G is not connected — contradiction!
2. If $d(v) = 1$: Let $G' = (V', E')$ be obtained by removing v and its edge. Then $|E'| = |V'| - 1$ and G' is still connected. By the induction hypothesis G' is a tree, and thus so is G. □