Computer Science 331

Trees, Spanning Trees, and Subgraphs

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Lecture #30

Outline

Computation of Spanning Trees

Motivation:

- Given a set of sites (represented by vertices of a graph), find paths connecting them all (or as many as possible) together.
- May be interested in cheapest possible connections (using connections represented by the edges of a weighted graph) or discovering which sites are reachable from any given site (search).

Goal for Today:

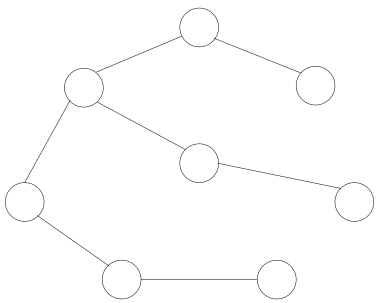
- Provide definitions and establish properties of trees and spanning trees required to solve these problems.

Reference:

- *Introduction to Algorithms*, Appendix B4 and B5

Trees

Definition: A *free tree* is a connected acyclic graph.



Frequently we just call a free tree a "tree."

- If we identify one vertex as the "root," then the result is the kind of "rooted tree" we have seen before.

Properties

We will present various properties and relations between $|V|$ and $|E|$ that characterize trees. Examples:

- If G is a tree then it has $|V| - 1$ edges
- An acyclic graph with $|V| - 1$ edges is a tree
- A connected graph with $|V| - 1$ edges is a tree

Reference: *Introduction to Algorithms*, Appendix B.5

Existence of Vertex With Degree At Most 1

Lemma 1

If $G = (V, E)$ is a graph such that $|V| \geq 2$ and $|E| < |V|$ then there exists a vertex $v \in V$ whose degree $d(v) \leq 1$.

Proof (by contradiction).

For any graph G, $\sum_{v \in V} d(v) = 2|E|$ (each edge counted twice)

If $d(v) \geq 2$ for every $v \in V$, then

$$2|E| = \sum_{v \in V} d(v) \geq \sum_{v \in V} 2 = 2|V|$$

so that $|E| \geq |V|$ — contradiction.

Thus, at least one vertex has degree at most one. □

Connected Graph has at Least $|V| - 1$ Edges

Lemma 2

If $G = (V, E)$ is connected then $|E| \geq |V| - 1$.

Proof (of contrapositive by induction on V).

Contrapositive: If $|E| < |V| - 1$ then G is not connected

Base case ($|V| = 1$): $|E| < |V| - 1 = 0$ implies G is not connected

Suppose $|V| \geq 2$ and $|E| < |V| - 1$. By Lemma 1, $\exists v$ with $d(v) \leq 1$.

1. If $d(v) = 0$: G is not connected (v has no edges).
2. If $d(v) = 1$: let $G' = (V', E')$ be obtained by removing v and its one edge (so $|E'| = |E| - 1$ and $|V'| = |V| - 1$).
   - $|E'| < |V'| - 1$, and by the induction hypothesis G' is not connected.
   - G is also not connected (adding vertex and one incident edge). □

Property of Cyclic Graphs

Lemma 3

If $G = (V, E)$ and each vertex $v \in V$ has degree at least two then G includes a cycle.

Proof.

Pick $v_1 \in V$, follow edges in E to reach $v_1, v_2, \ldots$ until either

1. some vertex appears for the second time, or
2. all edges incident to the current vertex have been used

Notice that:

- one of these cases must arise (because $|V|$ and $|E|$ are finite)
- if every $v \in V$ has $d(v) \geq 2$, then Case 1 occurs before Case 2

Thus, G includes a cycle. □

Acyclic Graph has at Most $|V| - 1$ Edges

Lemma 4

If $G = (V, E)$ is acyclic then $|E| \le |V| - 1$.

Proof (of contrapositive by induction on $|V|$).

□

A Tree has $|V| - 1$ Edges

Corollary 5

If $G = (V, E)$ is a tree then $|E| = |V| - 1$.

Proof.

□

Acyclic Graph with $|V| - 1$ Edges is a Tree

Lemma 6

If $G = (V, E)$ is acyclic and $|E| = |V| - 1$ then G is a tree.

Proof (induction on $|V|$).

□

Connected Graph with $|V| - 1$ Edges is a Tree

Lemma 7

If $G = (V, E)$ is connected and $|E| = |V| - 1$ then G is a tree.

Proof (induction on $|V|$).

□

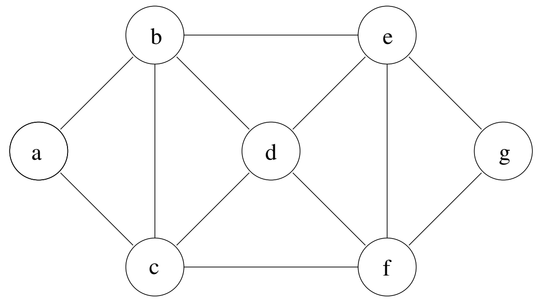
Spanning Trees

If $G = (V, E)$ is a connected undirected graph, then a *spanning tree* of G is a subgraph $\widehat{G} = (\widehat{V}, \widehat{E})$ of G such that

- $\widehat{V} = V$ (so that $\widehat{G}$ includes all the vertices in G)
- $\widehat{G}$ is a tree.

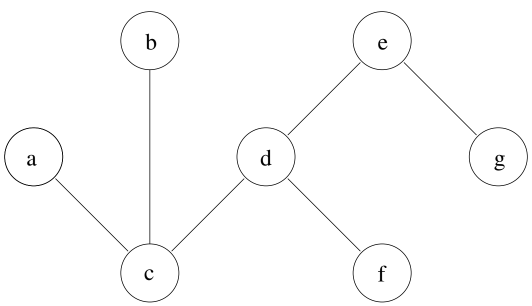
Example

Suppose $G = (V, E)$ is as follows.



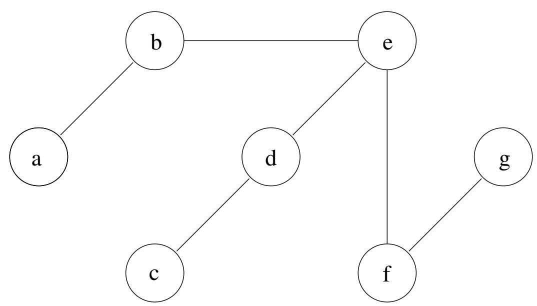
Example Tree 1

Is the following graph $G_1 = (V_1, E_1)$ a spanning tree of G?



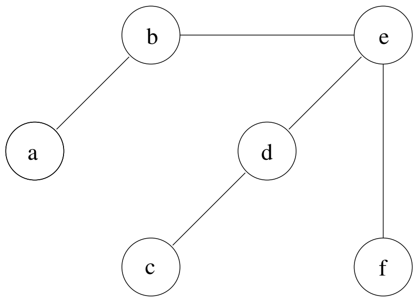
Example Tree 2

Is the following graph $G_2 = (V_2, E_2)$ is also a spanning tree of G?



Example Tree 3

Is the following graph $G_3 = (V_3, E_3)$ is also a spanning tree of G?



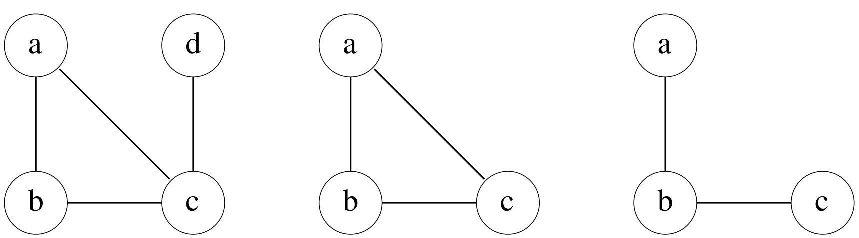
Subgraphs and Induced Subgraphs

Suppose $G = (V, E)$ is a graph.

- $\widehat{G} = (\widehat{V}, \widehat{E})$ is a *subgraph* of G if $\widehat{G}$ is a graph such that $\widehat{V} \subseteq V$ and $\widehat{E} \subseteq E$
- $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is an *induced subgraph* of G if
  - $\widetilde{G}$ is a subgraph of G and, furthermore
  - $\widetilde{E} = \left\{(u, v) \in E \mid u, v \in \widetilde{V}\right\}$, that is, $\widetilde{G}$ includes *all* the edges from G that it possibly could

Example

G_2 is an *induced subgraph* of G_1.
G_3 is a *subgraph* of G_1, but G_3 is **not** an *induced subgraph* of G_1.



G_1 G_2 G_3

Predecessor Subgraphs

Let $G = (V, E)$ be a graph and suppose there is a function $\pi : V \rightarrow V \cup \{\text{NIL}\}$ such that for some $s \in V$

- $V_p = \{s\} \cup \{v \in V \mid \pi(v) \neq \text{NIL}\}$
- $E_p = \{(\pi(v), v) \mid v \in V \text{ and } \pi(v) \neq \text{NIL}\}$
- $G_p = (V_p, E_p)$

We will require that subsequent algorithms construct G_p such that it is a subgraph of G and a tree.

Idea:

- $\pi(v)$ denotes the predecessor of v found by the algorithm
- collection of edges $(\pi(v), v)$ forms a spanning tree of G_p

Subgraph Property

Claim:

If, given G and $s \in V$, π is a function for which $G_p = (V_p, E_p)$ is as above, then

- *$\pi(v) \in V_p$ whenever $v \in V$ and $\pi(v) \neq NIL$ and*
- *$(\pi(v), v) \in E$ whenever $v \in V$ and $\pi(v) \neq NIL$*

Method of Proof.

Argue the two points in the Claim based on algorithm employed. □

Conclusion: G_p is a subgraph of G.

Tree Property

Claim:

If, given G and $s \in V$, π is a function for which $G_p = (V_p, E_p)$ is as above, then G_p is a tree.

Method of Proof.

Based on properties of the the algorithm:

- Argue that every vertex in V_p is reachable from s, implying that G_p is connected.
- $|E_p| = |V_p| - 1$, as the edges in E_p are $(\pi(v), v)$ for which $v \in V_p$ and $\pi(v) \neq NIL$
  - one edge for each $v \in V_p \setminus \{s\}$

By Lemma 7, G_p is a tree. □