CPSC 031 — Mathematics Review for CPSC 413

Exercise #1 — Mathematical Induction

September, 1999

Please try these exercises before the 6pm lecture on September 8.

1. Prove that $n^3 - n$ is divisible by 6 for every integer $n \geq 0$.

2. Find a positive constant c for which the following is true and prove your result.

$$\sum_{i=1}^{n} i^2 \leq c(n^3).$$

3. A *tree* is a connected graph with no cycles.

   A *planar map* is a graph embedded in the plane so that no two edges intersect except at endpoints.

   Prove the following **Theorem:** (Euler's Formula.) Let $G(V, E)$ be a planar map and let F be its set of faces. Then $|V| + |F| = |E| + 2$.

4. The *size* of a binary tree is the number of nodes in the tree.

   A node of a binary tree is an *internal node* if it has at least one child, and it is a *leaf* otherwise.

   A binary tree is a *complete* binary tree if every internal node has exactly two children.

   Prove that every nonempty complete binary tree has exactly one more leaf than it has internal nodes.

5. Lines in the plane are in *general position* if no two lines are parallel and no three lines intersect at a common point. Find the number of regions created by n lines in general position and prove your answer.

6. Prove that $n \geq 3$ lines in general position form at least $n - 2$ triangles.

7. An algorithm, *Convert-to-Binary*, is shown on the following page. Prove that when this algorithm terminates, the binary representation of the input n is stored in the array B.

Algorithm *Convert-to-Binary*

Input: A positive integer n

Output: An array B of bits corresponding to the binary representation of n.

```
begin
    t := n;
    k := 0;
    while t > 0 do
        k := k + 1;
        B[k] := t mod 2;
        t := t div 2;
    end while
end
```

Note that (a **mod** b) and (a **div** b) are, respectively, the remainder and quotient you'd get when you divide a positive integer a by a positive integer b, so that (a **mod** b) is always between 0 and $b - 1$, inclusive, and so that it's always true that

$$a = (a \textbf{ div } b) \times b + (a \textbf{ mod } b).$$